

# On a Kantorovich variant of $(p, q)$ -Szász-Mirakjan operators

<sup>1</sup>M. Mursaleen\*, <sup>1</sup>Khursheed J. Ansari and <sup>2</sup>Abylkassymova Elmira

<sup>1</sup>Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

<sup>2</sup>chair "The theory and methods of teaching informatics", Science-Pedagogical Faculty, M. Auezov South Kazakhstan State University, Tauke Khan Avenue 5, Shymkent 160012, Kazakhstan

mursaleenm@gmail.com; ansari.jkhursheed@gmail.com; smanchik\_armen@mail.ru

## Abstract

In the present paper we propose a Kantorovich variant of  $(p, q)$ -analogue of Szász-Mirakjan operators. We establish the moments of the operators with the help of a recurrence relation that we have derived and then prove the basic convergence theorem. Next, the local approximation as well as weighted approximation properties of these new operators in terms of modulus of continuity are studied.

**Keywords and phrases:**  $(p, q)$ -Szász-Mirakjan operators;  $(p, q)$ -Kantorovich-Szász-Mirakjan operators; modulus of continuity; weighted modulus of continuity;  $K$ -functional.

**AMS Subject Classifications (2010):** 41A10, 41A25, 41A36

## 1. Introduction and Notations

Approximation theory has been an established field of mathematics in the past century. The approximation of functions by positive linear operators is an important research topic in general mathematics and it also provides powerful tools to application areas such as computer-aided geometric design, numerical analysis, and solution of differential equations.

During the last two decades, the applications of  $q$ -calculus emerged as a new area in the field of approximation theory. The rapid development of  $q$ -calculus has led to the discovery of various generalizations of Bernstein polynomials involving  $q$ -integers. Several researchers introduced and studied many positive linear operators based on  $q$ -integers,  $q$ -Bernstein basis,  $q$ -Beta basis,  $q$ -derivative and  $q$ -integrals etc. Using  $q$ -integers, Lupaş [10] introduced the first  $q$ -Bernstein operators [4] and investigated its approximating and shape-preserving properties. Another  $q$ -analogue of the Bernstein polynomials is due to Phillips [16]. Since then several generalizations of well-known positive linear operators based on  $q$ -integers have been introduced and studied their approximation properties. Aral [2] and Aral and Gupta [3] proposed and studied some  $q$ -analogue of Szász-Mirakjan operators [18], defined on positive real axis. Also Mahmudov [11] introduced a  $q$ -parametric Szász-Mirakjan operators and studied their convergence properties. Recently, approximation properties for Kings type  $q$ -Bernstein-Kantorovich operators have been studied in [15].

Very recently, Mursaleen et al applied  $(p, q)$ -calculus in approximation theory and introduced the  $(p, q)$ -analogue of Bernstein operators [12, 13] and  $(p, q)$ -Bernstein-Stancu operators [14] and investigated their approximation properties. Also Acar [1] has introduced  $(p, q)$  parametric generalization of Szász-Mirakjan operators. In the present work we have proposed a Kantorovich variant of Szász-Mirakjan operators and establish the moments of the operators with the help of a recurrence relation that we have derived and then prove the basic convergence theorem. Next, the local approximation as well as weighted approximation properties of these new operators in terms of modulus of continuity are studied.

---

\*Corresponding author

The  $(p, q)$ -integer was introduced in order to generalize or unify several forms of  $q$ -oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [5]. Let us recall certain notations of  $(p, q)$ -calculus:

The  $(p, q)$ -integers  $[n]_{p,q}$  are defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \dots, \quad 0 < q < p \leq 1.$$

The  $(p, q)$ -factorial and  $(p, q)$ -Binomial coefficients are defined by

$$[n]_{p,q}! := \begin{cases} [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}, & n \in \mathbb{N}; \\ 1, & n = 0 \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},$$

respectively. Further, the  $(p, q)$ -binomial expansions are given as

$$(ax + by)_{p,q}^n := \sum_{k=0}^n p^{\binom{n-k}{2}} q^{\binom{k}{2}} a^{n-k} b^k x^{n-k} y^k$$

and

$$(x - y)_{p,q}^n := (x - y)(px - qy)(p^2x - q^2y) \cdots (p^{n-1}x - q^{n-1}y).$$

Let  $m$  and  $n$  be two non-negative integers. Then the following assertion is valid

$$(x - y)_{p,q}^{m+n} := (x - y)_{p,q}^m (p^m x - q^m y)_{p,q}^n.$$

Also, the  $(p, q)$ -derivative of a function  $f$ , denoted by  $D_{p,q}f$ , is defined by

$$(D_{p,q}f)(x) := \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0, \quad (D_{p,q}f)(0) := f'(0)$$

provided that  $f$  is differentiable at 0. The  $(p, q)$ -derivative fulfils the following product rules

$$\begin{aligned} D_{p,q}(f(x)g(x)) &:= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x), \\ D_{p,q}(f(x)g(x)) &:= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x). \end{aligned}$$

Moreover,

$$\begin{aligned} D_{p,q}\left(\frac{f(x)}{g(x)}\right) &:= \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}, \\ D_{p,q}\left(\frac{f(x)}{g(x)}\right) &:= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}. \end{aligned}$$

We consider the  $(p, q)$ -exponential functions in the following forms:

$$\begin{aligned} e_{p,q}(x) &= \sum_{n=0}^{\infty} p^{n(n-1)/2} \frac{x^n}{[n]_{p,q}!}, \\ E_{p,q}(x) &= \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_{p,q}!}, \end{aligned}$$

which satisfy the equality  $e_{p,q}(x)E_{p,q}(-x) = 1$ . The definite integrals of the function  $f$  are defined by

$$\int_0^a f(x) d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right), \quad \text{when} \quad \left|\frac{p}{q}\right| < 1,$$

and

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right), \quad \text{when} \quad \left|\frac{p}{q}\right| > 1.$$

Details on  $(p, q)$ -calculus can be found in [5, 8, 17]. For  $p = 1$ , all the notions of  $(p, q)$ -calculus are reduced to  $q$ -calculus.

## 2. Operators and estimation of moments

Now we set the  $(p, q)$ -Szász-Mirakjan basis function as

$$s_n(p, q; x) =: E_{p,q}\left(-[n]_{p,q}x\right) \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!}.$$

For  $q \in (0, p)$ ,  $p \in (q, 1]$  and  $x \in [0, \infty)$ ,  $s_n(p, q; x) \geq 0$ . We can easily check that

$$\sum_{k=0}^{\infty} s_n(p, q; x) =: E_{p,q}\left(-[n]_{p,q}x\right) \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} = 1.$$

For  $0 < q < p \leq 1$  the  $(p, q)$ -Szász-Mirakjan operators are defined as

$$S_n(f, p, q; x) = [n]_{p,q} \sum_{k=0}^{\infty} p^{-k} q^k s_{n,k}(p, q; x) f\left(\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right), \quad x \in [0, \infty). \quad (1)$$

From the definition of the  $(p, q)$ -Szász-Mirakjan operators we derive the following formulas.

**Lemma 1.** Let  $0 < q < p \leq 1$ . We have

- (i)  $S_n(1, p, q; x) = 1$ ;
- (ii)  $S_n(t, p, q; x) = x$ ;
- (iii)  $S_n(t^2, p, q; x) = \frac{px^2}{q} + \frac{x}{[n]_{p,q}}$ ;
- (iv)  $S_n(t^3, p, q; x) = \frac{p^3}{q^3}x^3 + \frac{p^2+2pq}{q[n]_{p,q}}x^2 + \frac{q^2}{[n]_{p,q}^2}x$ ;
- (v)  $S_n(t^4, p, q; x) = \frac{p^6}{q^6}x^4 + \frac{p^3(p^2+2q+3q^2)}{q^4[n]_{p,q}}x^3 + \frac{p(p^2+3pq+3q^2)}{q[n]_{p,q}}x^2 + \frac{q^3}{[n]_{p,q}^3}x$ .

Now we propose our Kantorovich variant of  $(p, q)$ -Szász-Mirakjan operators (1) as follows:

For  $f \in C[0, \infty)$ ,  $0 < q < p \leq 1$  and each positive integer  $n$ ,

$$K_n(f, p, q; x) = [n]_{p,q} \sum_{k=0}^{\infty} p^{-k} q^k s_{n,k}(p, q; x) \int_{\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}} f(t) d_{p,q}t \quad (2)$$

We will derive the recurrence formula for  $K_n(t^m, p, q; x)$  and calculate  $K_n(t^m, p, q; x)$  for  $m = 0, 1, 2$ .

**Lemma 2.** For the operators  $K_n$  we have

$$K_n(t^m, p, q; x) = \frac{1}{[m+1]_{p,q}} \sum_{j=0}^m \sum_{i=0}^j \frac{p^i}{q^i [n]_{p,q}^{j-i}} \binom{j}{i} S_n(t^{m+i-j}, p, q; x). \quad (3)$$

**Proof.** Using the expansion  $a^{m+1} - b^{m+1} = (a-b)(a^m + a^{m-1}b + \dots + ab^{m-1} + b^m)$  we have

$$\int_{\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}} t^m d_{p,q}t = \frac{1}{[m+1]_{p,q}} \left\{ \left( \frac{[k+1]_{p,q}}{q^k[n]_{p,q}} \right)^{m+1} - \left( \frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}} \right)^{m+1} \right\}.$$

Using  $[k+1]_{p,q} = p^k + q[k]_{p,q}$  and also  $[k+1]_{p,q} = q^k + p[k]_{p,q}$ , we have

$$\begin{aligned} \int_{\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}} t^m d_{p,q}t &= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k[n]_{p,q}} \sum_{j=0}^m \left( \frac{[k+1]_{p,q}}{q^k[n]_{p,q}} \right)^j \left( \frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}} \right)^{m-j} \\ &= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k[n]_{p,q}} \sum_{j=0}^m \left( \frac{q^k + p[k]_{p,q}}{q^k[n]_{p,q}} \right)^j \left( \frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}} \right)^{m-j} \\ &= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k[n]_{p,q}} \sum_{j=0}^m \sum_{i=0}^j \binom{j}{i} \frac{p^i [k]_{p,q}^i q^{k(j-i)}}{q^{kj} [n]_{p,q}^j} \frac{[k]_{p,q}^{m-j}}{q^{(k-1)(m-j)} [n]_{p,q}^{m-j}} \\ &= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k[n]_{p,q}} \sum_{j=0}^m \sum_{i=0}^j \binom{j}{i} \frac{p^i [k]_{p,q}^{m+i-j}}{q^{ki} [n]_{p,q}^m q^{(k-1)(m-j)}}. \end{aligned}$$

Writing this in the definition of  $K_n(t^m, p, q; x)$ , we get

$$\begin{aligned} K_n(t^m, p, q; x) &= [n]_{p,q} \sum_{k=0}^{\infty} p^{-k} q^k s_{n,k}(p, q; x) \int_{\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}} t^m d_{p,q}t \\ &= \frac{1}{[m+1]_{p,q}} \sum_{j=0}^m \sum_{k=0}^{\infty} s_{n,k}(p, q; x) \sum_{i=0}^j \frac{p^i}{q^i [n]_{p,q}^{j-i}} \binom{j}{i} \frac{[k]_{p,q}^{m+i-j}}{q^{(k-1)(m+i-j)} [n]_{p,q}^{m+i-j}} \\ &= \frac{1}{[m+1]_{p,q}} \sum_{j=0}^m \sum_{i=0}^j \frac{p^i}{q^i [n]_{p,q}^{j-i}} \binom{j}{i} \sum_{k=0}^{\infty} \frac{[k]_{p,q}^{m+i-j}}{q^{(k-1)(m+i-j)} [n]_{p,q}^{m+i-j}} s_{n,k}(p, q; x) \\ &= \frac{1}{[m+1]_{p,q}} \sum_{j=0}^m \sum_{i=0}^j \frac{p^i}{q^i [n]_{p,q}^{j-i}} \binom{j}{i} S_n(t^{m+i-j}, p, q; x). \end{aligned}$$

Using the recurrence formula (3) we may easily calculate  $K_n(t^m, p, q; x)$  for  $m = 0, 1, 2$ .

**Lemma 3.** We have

- (i)  $K_n(1, p, q; x) = 1$ ;
- (ii)  $K_n(t, p, q; x) = \frac{1}{q}x + \frac{1}{[2]_{p,q}[n]_{p,q}}$ ;

$$\begin{aligned}
\text{(iii)} \quad K_n(t^2, p, q; x) &= \frac{p}{q^3}x^2 + \left( \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} \right)x + \frac{1}{[3]_{p,q}[n]_{p,q}^2}; \\
\text{(iv)} \quad K_n(t^3, p, q; x) &= \frac{p^3}{q^6}x^3 + \left( \frac{p^2+2pq}{q^4[n]_{p,q}} + \frac{p(3p^2+2pq+q^2)}{q^3[4]_{p,q}[n]_{p,q}} \right)x^2 + \left( \frac{1}{q[n]_{p,q}^2} + \frac{3p^2+2pq+q^2}{q^2[4]_{p,q}[n]_{p,q}^2} + \frac{3p+q}{q[4]_{p,q}[n]_{p,q}^2} \right)x + \frac{1}{[4]_{p,q}[n]_{p,q}^3}; \\
\text{(v)} \quad K_n(t^4, p, q; x) &= \frac{p^6}{q^{10}}x^4 + \left( \frac{p^3(p^2+2q+3q^2)}{q^8[n]_{p,q}} + \frac{p^3(4p^3+3p^2q+2pq^2+q^3)}{q^6[5]_{p,q}[n]_{p,q}} \right)x^3 + \left( \frac{p(p^2+3pq+3q^2)}{q^5[n]_{p,q}^2} + \frac{(p^2+2pq)(4p^3+3p^2q+2pq^2+q^3)}{q^4[5]_{p,q}[n]_{p,q}^2} + \frac{p(6p^2+3pq+q^2)}{q^3[5]_{p,q}[n]_{p,q}^2} \right)x^2 + \left( \frac{1}{q[n]_{p,q}^3} + \frac{4p^3+3p^2q+2pq^2+q^3}{q[5]_{p,q}[n]_{p,q}^3} + \frac{6p^2+3pq+q^2}{q^2[5]_{p,q}[n]_{p,q}^3} + \frac{4p+q}{q[5]_{p,q}[n]_{p,q}^3} \right)x + \frac{1}{[5]_{p,q}[n]_{p,q}^4}; \\
\text{(iv)} \quad K_n((t-x), p, q; x) &= \frac{1-q}{q}x + \frac{1}{[2]_{p,q}[n]_{p,q}}; \\
\text{(v)} \quad K_n((t-x)^2, p, q; x) &= \left( \frac{p}{q^3} - \frac{2}{q} + 1 \right)x^2 + \left( \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} - \frac{2}{[2]_{p,q}[n]_{p,q}} \right)x + \frac{1}{[3]_{p,q}[n]_{p,q}^2}; \\
\text{(vi)} \quad K_n((t-x)^4, p, q; x) &= x^4 \left( \frac{p^6}{q^{10}} - \frac{4p^3}{q^6} + \frac{6p}{q^3} - \frac{4}{q} + 1 \right) + \frac{x^3}{[n]_{p,q}} \left( \frac{p^3(p^2+2q+3q^2)}{q^8} + \frac{p^3(4p^3+3p^2q+2pq^2+q^3)}{q^6[5]_{p,q}} - \frac{4(p^2+2pq)}{q^4} - \frac{4p(3p^2+2pq+q^2)}{q^3[4]_{p,q}} + \frac{6(2p+q)}{q[3]_{p,q}} + \frac{6}{q^2} - \frac{4}{[2]_{p,q}} \right) + \frac{x^2}{[n]_{p,q}^2} \left( \frac{p(p^2+3pq+3q^2)}{q^5} + \frac{(p^2+2pq)(4p^3+3p^2q+2pq^2+q^3)}{q^4[5]_{p,q}} + \frac{p(6p^2+3pq+q^2)}{q^3[5]_{p,q}} - \frac{4(3p^2+2pq+q^2)}{q^2} - \frac{4}{q} + \frac{6}{[3]_{p,q}} \right) + \frac{x}{[n]_{p,q}^3} \left( \frac{1}{q} + \frac{4p^3+3p^2q+2pq^2+q^3}{q[5]_{p,q}} + \frac{6p^2+3pq+q^2}{q^2[5]_{p,q}} + \frac{4p+q}{q[5]_{p,q}} \right).
\end{aligned} \tag{4}$$

**Proof.** Obviously, with the help of Lemma 1, we can get

$$\begin{aligned}
K_n(t, p, q; x) &= \frac{1}{[2]_{p,q}} \left\{ \left( 1 + \frac{p}{q} \right) S_n(t, p, q; x) + \frac{1}{[n]_{p,q}} S_n(1, p, q; x) \right\} \\
&= \frac{1}{q}x + \frac{1}{[2]_{p,q}[n]_{p,q}},
\end{aligned}$$

$$\begin{aligned}
K_n(t^2, p, q; x) &= \frac{1}{[3]_{p,q}} \left\{ \left( 1 + \frac{p}{q} + \frac{p^2}{q^2} \right) S_n(t^2, p, q; x) + \left( \frac{1}{[n]_{p,q}} + \frac{2p}{q[n]_{p,q}} \right) S_n(t, p, q; x) + \frac{1}{[n]_{p,q}^2} S_n(1, p, q; x) \right\} \\
&= \frac{1}{q^2} S_n(t^2, p, q; x) + \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} S_n(t, p, q; x) + \frac{1}{[3]_{p,q}[n]_{p,q}^2} S_n(1, p, q; x) \\
&= \frac{p}{q^3}x^2 + \left( \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} \right)x + \frac{1}{[3]_{p,q}[n]_{p,q}^2}.
\end{aligned}$$

Using the linearity of the operators, we can have

$$\begin{aligned}
K_n((t-x)^2, p, q; x) &= K_n(t^2, p, q; x) - 2xK_n(t, p, q; x) + x^2K_n(1, p, q; x) \\
&= \frac{p}{q^3}x^2 + \left( \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} \right)x + \frac{1}{[3]_{p,q}[n]_{p,q}^2} - 2x \left( \frac{1}{q}x + \frac{1}{[2]_{p,q}[n]_{p,q}} \right) + x^2 \\
&= \left( \frac{p}{q^3} - \frac{2}{q} + 1 \right)x^2 + \left( \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} - \frac{2}{[2]_{p,q}[n]_{p,q}} \right)x + \frac{1}{[3]_{p,q}[n]_{p,q}^2}.
\end{aligned}$$

**Remark** For  $q \in (0, 1)$  and  $p \in (q, 1]$  it is obvious that  $\lim_{n \rightarrow \infty} [n]_{p,q} = \frac{1}{p-q}$ . In order to reach to convergence results of the operator  $K_n$  we take sequences  $q_n \in (0, 1)$  and  $p_n \in (q_n, 1]$  such that  $\lim_{n \rightarrow \infty} p_n = 1$   $\lim_{n \rightarrow \infty} q_n = 1$ . So we get that  $\lim_{n \rightarrow \infty} [n]_{p_n, q_n} = \infty$ .

Thus the above remark provides an example that such a sequence can always be constructed. If we choose for  $a > b > 0$ ,  $q_n = \frac{n}{n+a} < \frac{n}{n+b} = p_n$  such that  $0 < q_n < p_n \leq 1$ , it can be easily seen that  $\lim_{n \rightarrow \infty} p_n = 1$ ,  $\lim_{n \rightarrow \infty} q_n = 1$  and  $\lim_{n \rightarrow \infty} p_n^n = e^{-b}$ ,  $\lim_{n \rightarrow \infty} q_n^n = e^{-a}$ . Hence we guarantee that  $\lim_{n \rightarrow \infty} [n]_{p_n, q_n} = \infty$ .

### 3. Direct approximation result

In this section we study the Korovkin's approximation property of the Kantorovich variant of  $(p, q)$ -Szász operators.

**Theorem 4.** Let  $0 < q_n < p_n \leq 1$  and  $A > 0$ . Then for each  $f \in C_m[0, \infty) := \{f \in C[0, \infty) : |f(x)| \leq M_f(1+x^m), \text{ for some } M_f > 0 \text{ depending on } f, m > 0\}$  where  $C_m[0, \infty)$  be endowed with the norm  $\|f\|_m = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^m}$ , the sequence of operators  $K_n(f, p_n, q_n; x)$  converges to  $f$  uniformly on  $[0, A]$  if and only if  $\lim_{n \rightarrow \infty} p_n = 1$  and  $\lim_{n \rightarrow \infty} q_n = 1$ .

**Proof.** First, we assume that  $\lim_{n \rightarrow \infty} p_n = 1$  and  $\lim_{n \rightarrow \infty} q_n = 1$ . Now, we have to show that  $K_n(f, p_n, q_n; x)$  converges to  $f$  uniformly on  $[0, A]$ .

From Lemma 3, we see that

$$K_n(1, p_n, q_n; x) \rightarrow 1, \quad K_n(t, p_n, q_n; x) \rightarrow x, \quad K_n(t^2, p_n, q_n; x) \rightarrow x^2,$$

uniformly on  $[0, A]$  as  $n \rightarrow \infty$ .

Therefore, the well-known property of the Korovkin theorem implies that  $K_n(f, p_n, q_n; x)$  converges to  $f$  uniformly on  $[0, A]$  provided  $f \in C_m[0, \infty)$ .

We show the converse part by contradiction. Assume that  $p_n$  and  $q_n$  do not converge to 1. Then they must contain subsequences  $p_{n_k} \in (0, 1)$ ,  $q_{n_k} \in (0, 1)$ ,  $p_{n_k} \rightarrow a \in [0, 1)$  and  $q_{n_k} \rightarrow b \in [0, 1)$  as  $k \rightarrow \infty$ , respectively.

Thus,

$$\frac{1}{[n_k]_{p_{n_k}, q_{n_k}}} = \frac{p_{n_k} - q_{n_k}}{(p_{n_k})^{n_k} - (q_{n_k})^{n_k}} \rightarrow 0 \text{ as } k \rightarrow \infty$$

and we get

$$K_n(t, p_{n_k}, q_{n_k}; x) - x = \frac{1}{q_{n_k}}x + \frac{1}{[2]_{p_{n_k}, q_{n_k}}[n]_{p_{n_k}, q_{n_k}}} - x \rightarrow \frac{x}{b} - x \neq 0.$$

This leads to a contradiction. Thus  $p_n \rightarrow 1$  and  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ .

**Theorem 5.** Let  $f \in C_2[0, \infty)$ ,  $q = q_n \in (0, 1)$  and  $p = p_n \in (q, 1]$  such that  $p_n \rightarrow 1$ ,  $q_n \rightarrow 1$  as  $n \rightarrow \infty$  and  $\omega_{a+1}(f, \delta)$  be the modulus of continuity on the finite interval  $[0, a+1] \subset [0, \infty)$ , where  $a > 0$ . Then

$$|K_n(f, p, q; x) - f(x)| \leq 4M_f(1+a^2)\delta_n^2(x) + 2\omega_{a+1}(f, \delta_n(x))$$

where  $\delta_n(x) = \sqrt{K_n((t-x)^2, p, q; x)}$ , given by (4).

**Proof.** For  $x \in [0, a]$  and  $t > a + 1$ , since  $t - x > 1$ , we have

$$|f(t) - f(x)| \leq M_f(2 + x^2 + t^2) \leq M_f(2 + 3x^2 + 2(t - x)^2) \leq M_f(4 + 3x^2)(t - x)^2 \leq 4M_f(1 + a^2)(t - x)^2. \quad (5)$$

For  $x \in [0, a]$  and  $t \leq a + 1$ , we have

$$|f(t) - f(x)| \leq \omega_{a+1}(f, |t - x|) \leq \left(1 + \frac{|t - x|}{\delta}\right) \omega_{a+1}(f, \delta) \quad (6)$$

with  $\delta > 0$ .

From (5) and (6), we may write

$$|f(t) - f(x)| \leq 4M_f(1 + a^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{a+1}(f, \delta), \quad (7)$$

for  $x \in [0, a]$  and  $t \geq 0$ . Thus, by applying the Cauchy-Schwarz's inequality, we have

$$\begin{aligned} |K_n(f, p, q; x) - f(x)| &\leq K_n(|f(t) - f(x)|, p, q; x) \\ &\leq 4M_f(1 + a^2)K_n((t - x)^2, p, q; x) + \left(1 + \frac{1}{\delta} \sqrt{K_n((t - x)^2, p, q; x)}\right) \omega_{a+1}(f, \delta) \\ &\leq 4M_f(1 + a^2)\delta_n^2(x) + 2\omega_{a+1}(f, \delta_n(x)) \end{aligned}$$

on choosing  $\delta := \delta_n(x)$ . This completes the proof of the theorem.

#### 4. Local approximation

In this section we establish local approximation theorem for the Kantorovich variant of  $(p, q)$ -Szász operators. Let  $C_B[0, \infty)$  be the space of all real-valued continuous bounded functions  $f$  on  $[0, \infty)$ , endowed with the norm  $\|f\| = \sup_{x \in [0, \infty)} |f(x)|$ . The Peetre's K-functional is defined by

$$K_2(f, \delta) = \inf_{g \in C^2[0, \infty)} \{\|f - g\| + \delta \|g''\|\},$$

where  $C_B^2[0, \infty) := \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$ . By [4, p.177, Theorem 2.4], there exists an absolute constant  $M > 0$  such that

$$K_2(f, \delta) \leq M\omega_2(f, \sqrt{\delta}), \quad (8)$$

where  $\delta > 0$  and the second order modulus of smoothness is defined as

$$\omega_2(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x + 2h) - 2f(x + h) + f(x)|, \quad (9)$$

where  $f \in C_B[0, \infty)$  and  $\delta > 0$ .

**Theorem 6.** Let  $f \in C_B[0, \infty)$  and  $0 < q < p \leq 1$ . Then, for every  $x \in [0, \infty)$ , we have

$$|K_n(f, p, q; x) - f(x)| \leq M\omega_2\left(f, \sqrt{\delta_n(x)}\right) + \omega\left(f, \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1 - q}{q}x\right),$$

where  $M$  is an absolute constant and

$$\delta_n(x) = K_n((t-x)^2, p, q; x) + \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1-q}{q}x \right)^2.$$

**Proof.** For  $x \in [0, \infty)$ , we consider the auxiliary operators  $K_n^*$  defined by

$$K_n^*(f, p, q; x) = K_n(f, p, q; x) - f\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x\right) + f(x). \quad (10)$$

From Lemma 3, we observe that the operators  $K_n^*(f, p, q; x)$  are linear and reproduce the linear functions. Hence

$$\begin{aligned} K_n^*((t-x), p, q; x) &= K_n((t-x), p, q; x) - \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - x \right) \\ &= K_n(t, p, q; x) - xK_n(1, p, q; x) - \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x \right) + x = 0. \end{aligned} \quad (11)$$

Let  $x \in [0, \infty)$  and  $g \in C_B^2[0, \infty)$ . Using the Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u) g''(u) du.$$

Applying  $K_n^*$  to both sides of the above equation and using (11), we have

$$\begin{aligned} K_n^*(g, p, q; x) - g(x) &= K_n^*((t-x)g'(x), p, q; x) + K_n^*\left(\int_x^t (t-u)g''(u)du, p, q; x\right) \\ &= g'(x)K_n^*((t-x), p, q; x) + K_n^{(p,q)}\left(\int_x^t (t-u)g''(u)du, p, q; x\right) \\ &\quad - \int_x^{\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x} \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u \right) g''(u)du \\ &= K_n\left(\int_x^t (t-u)g''(u)du, p, q; x\right) \\ &\quad - \int_x^{\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x} \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u \right) g''(u)du. \end{aligned}$$

On the other hand, since

$$\left| \int_x^t (t-u)g''(u)du \right| \leq \int_x^t |t-u||g''(u)|du \leq \|g''\| \int_x^t |t-u|du \leq (t-x)^2 \|g''\|$$

and

$$\begin{aligned} &\left| \int_x^{\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x} \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u \right) g''(u)du \right| \\ &\leq \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u \right)^2 \|g''\| \end{aligned}$$



we conclude that

$$\begin{aligned}
|K_n^*(g, p, q; x) - g(x)| &= \left| K_n \left( \int_x^t (t-u) g''(u) du, p, q; x \right) \right. \\
&\quad \left. - \int_x^{\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x} \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u \right) g''(u) du \right| \\
&\leq \|g''\| K_n((t-x)^2, p, q; x) + \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - x \right)^2 \|g''\| \\
&= \delta_n(x) \|g''\|.
\end{aligned} \tag{12}$$

Now, taking into account boundedness of  $K_n^*$  by (10), we have

$$|K_n^*(f, p, q; x)| \leq |K_n(f, p, q; x)| + 2\|f\| \leq 3\|f\| \tag{13}$$

Using (12) and (13) in (10), we obtain

$$\begin{aligned}
|K_n(f, p, q; x) - f(x)| &\leq |K_n^*(f, p, q; x) - f(x)| + \left| f(x) - f \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x \right) \right| \\
&\leq |K_n^*(f - g, p, q; x) - (f - g)(x)| \\
&\quad + \left| f(x) - f \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x \right) \right| + |K_n^*(g, p, q; x) - g(x)| \\
&\leq |K_n^*(f - g, p, q; x)| + |(f - g)(x)| \\
&\quad + \left| f(x) - f \left( \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x \right) \right| + |K_n^*(g, p, q; x) - g(x)| \\
&\leq 4\|f - g\| + \omega \left( f, \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1-q}{q}x \right) + \delta_n(x) \|g''\|.
\end{aligned}$$

Hence, taking the infimum on the right-hand side over all  $g \in C_B^2[0, \infty)$ , we have the following result

$$|K_n(f, p, q; x) - f(x)| \leq 4K_2(f, \delta_n(x)) + \omega \left( f, \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1-q}{q}x \right).$$

In view of the property of  $K$ -functional (8), we get

$$|K_n(f, p, q; x) - f(x)| \leq M\omega_2 \left( f, \sqrt{\delta_n(x)} \right) + \omega \left( f, \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1-q}{q}x \right).$$

This completes the proof of the theorem.

## 5. Weighted approximation

Let  $f \in C_2^*[0, \infty) := \{f \in C_2[0, \infty) : \lim_{x \rightarrow \infty} \frac{|f(x)|}{1+x^2} < \infty\}$ . Throughout the section, we assume that  $(p_n)$  and  $(q_n)$  are sequences such that  $0 < q_n < p_n \leq 1$  and  $p_n \rightarrow 1$ ,  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ .

**Theorem 7.** For each  $f \in C_2^*[0, \infty)$ , we have

$$\lim_{n \rightarrow \infty} \|K_n(f, p_n, q_n; x) - f(x)\|_2 = 0.$$

**Proof.** Using the Korovkin type theorem on weighted approximation in [7] we see that it is sufficient to verify the following three conditions

$$\lim_{n \rightarrow \infty} \|K_n(t^i, p_n, q_n; x) - x^i\|_2 = 0, \quad i = 0, 1, 2. \quad (14)$$

Since  $K_n(1, p_n, q_n; x) = 1$ , (14) holds true for  $m = 0$ .

By Lemma 3, we have

$$\begin{aligned} \|K_n(t, p_n, q_n; x) - x\|_2 &= \sup_{x \in [0, \infty)} \frac{|K_n(t, p_n, q_n; x) - x|}{1 + x^2} \\ &= \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \left| \frac{1}{q_n} x + \frac{1}{[2]_{p,q}[n]_{p,q}} - x \right| \\ &\leq \left( \frac{1}{q_n} - 1 \right) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} + \frac{1}{[2]_{p,q}[n]_{p,q}} \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \\ &\leq \frac{1}{q_n} - 1 + \frac{1}{[2]_{p,q}[n]_{p,q}}. \end{aligned}$$

which implies that the condition in (14) holds for  $i = 1$  as  $n \rightarrow \infty$ .

Similarly we can write

$$\begin{aligned} \|K_n(t^2, p_n, q_n; x) - x^2\|_2 &= \sup_{x \in [0, \infty)} \frac{|K_n(t^2, p_n, q_n; x) - x^2|}{1 + x^2} \\ &\leq \left( \frac{p_n}{q_n^3} - 1 \right) \sup_{x \in [0, \infty)} \frac{x^2}{1 + x^2} \\ &\quad + \left( \frac{2p_n + q_n}{q_n[3]_{p_n, q_n}[n]_{p_n, q_n}} + \frac{1}{q_n^2[n]_{p_n, q_n}} \right) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} \\ &\quad + \frac{1}{[3]_{p_n, q_n}[n]_{p_n, q_n}^2} \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \\ &\leq \frac{p_n}{q_n^3} - 1 + \frac{2p_n + q_n}{q_n[3]_{p_n, q_n}[n]_{p_n, q_n}} + \frac{1}{q_n^2[n]_{p_n, q_n}} + \frac{1}{[3]_{p_n, q_n}[n]_{p_n, q_n}^2}, \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \|K_n(t^2, p_n, q_n; x) - x^2\|_2 = 0,$$

(14) holds for  $i = 2$ . Thus the proof is completed.

We give the following theorem to approximate all functions in  $C_2^*[0, \infty)$ . This type of results are given in [6] for classical Szász operators.

**Theorem 8.** For each  $f \in C_2^*[0, \infty)$  and  $\alpha > 0$ , we have

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, \infty)} \frac{|K_n(f, p_n, q_n; x) - f(x)|}{(1 + x^2)^{1+\alpha}} = 0.$$

**Proof.** Let  $x_0 \in [0, \infty)$  be arbitrary but fixed. Then

$$\begin{aligned} \sup_{x \in [0, \infty)} \frac{|K_n(f, p_n, q_n; x) - f(x)|}{(1+x^2)^{1+\alpha}} &= \sup_{x \leq x_0} \frac{|K_n(f, p_n, q_n; x) - f(x)|}{(1+x^2)^{1+\alpha}} + \sup_{x > x_0} \frac{|K_n(f, p_n, q_n; x) - f(x)|}{(1+x^2)^{1+\alpha}} \\ &\leq \|K_n(f) - f\|_{C[0, x_0]} + \|f\|_2 \sup_{x > x_0} \frac{|K_n(1+t^2, p, q; x)|}{(1+x^2)^{1+\alpha}} + \sup_{x > x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}}. \end{aligned} \quad (15)$$

Since  $|f(x)| \leq \|f\|_2(1+x^2)$ , we have  $\sup_{x > x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}} \leq \frac{\|f\|_2}{(1+x_0^2)^\alpha}$ .

Let  $\varepsilon > 0$  be arbitrary. We can choose  $x_0$  to be so large that

$$\frac{\|f\|_2}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}. \quad (16)$$

In view of Theorem 4, we obtain

$$\|f\|_2 \lim_{n \rightarrow \infty} \frac{|K_n(1+t^2, p, q; x)|}{(1+x^2)^{1+\alpha}} = \frac{1+x^2}{(1+x^2)^{1+\alpha}} \|f\|_2 = \frac{\|f\|_2}{(1+x^2)^\alpha} \leq \frac{\|f\|_2}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}. \quad (17)$$

Using Theorem 5, we can see that the first term of the inequality (15), implies that

$$\|K_n(f) - f\|_{C[0, x_0]} < \frac{\varepsilon}{3}, \quad \text{as } n \rightarrow \infty. \quad (18)$$

Combining (16)-(18), we get that desired result.

For  $f \in C_2^*[0, \infty)$ , the weighted modulus of continuity is defined as

$$\Omega_2(f, \delta) = \sup_{x \geq 0, 0 < h \leq \delta} \frac{|f(x+h) - f(x)|}{1+(x+h)^2}.$$

**Lemma 9** ([9]). If  $f \in C_2^*[0, \infty)$  then

- (i)  $\Omega_2(f, \delta)$  is monotone increasing function of  $\delta$ ,
- (ii)  $\lim_{\delta \rightarrow 0^+} \Omega_2(f, \delta) = 0$ ,
- (iii) for any  $\lambda \in [0, \infty)$ ,  $\Omega_2(f, \lambda\delta) \leq (1+\lambda)\Omega_2(f, \delta)$ .

**Theorem 10.** If  $f \in C_2^*[0, \infty)$ , then for sufficiently large  $n$  we have

$$|K_n(f, p, q; x) - f(x)| \leq K(1+x^{2+\lambda})\Omega_2(f, \delta_n), \quad x \in [0, \infty),$$

where  $\lambda \geq 1$ ,  $\delta_n = \max\{\alpha_n, \beta_n, \gamma_n\}$ ,  $\alpha_n, \beta_n, \gamma_n$  being

$$\alpha_n = \frac{p}{q^3} - \frac{2}{q} + 1, \quad \beta_n = \frac{p + [2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} - \frac{2}{[2]_{p,q}[n]_{p,q}}, \quad \gamma_n = \frac{1}{[3]_{p,q}[n]_{p,q}^2}.$$

**Proof.** From the definition of  $\Omega_2(f, \delta)$  and Lemma 9, we may write

$$\begin{aligned} |f(t) - f(x)| &\leq (1 + (x + |t - x|)^m) \left( \frac{|t - x|}{\delta} + 1 \right) \Omega_2(f, \delta) \\ &\leq (1 + (2x + t)^m) \left( \frac{|t - x|}{\delta} + 1 \right) \Omega_2(f, \delta) \\ &:= \varphi_x(t) \left( 1 + \frac{1}{\delta} \psi_x(t) \right) \Omega_2(f, \delta). \end{aligned}$$

Then we obtain

$$|K_n(f, p, q; x) - f(x)| \leq \Omega_2(f, \delta_n) \left( K_n(\varphi_x, p, q; x) + \frac{1}{\delta_n} K_n(\varphi_x \psi_x, p, q; x) \right).$$

Applying the Cauchy-Schwartz inequality to the second term on the right-hand side, we get

$$|K_n(f, p, q; x) - f(x)| \leq \Omega_2(f, \delta) \left( K_n(\varphi_x, p, q; x) + \frac{1}{\delta_n} \sqrt{K_n(\varphi_x^2, p, q; x)} \sqrt{K_n(\psi_x^2, p, q; x)} \right). \quad (19)$$

From Lemma 3, we get

$$\begin{aligned} \frac{1}{1+x^2} K_n(1+t^2, p, q; x) &= \frac{1}{1+x^2} + \frac{p}{q^3} \frac{x^2}{1+x^2} + \left( \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} \right) \frac{x}{1+x^2} + \frac{1}{[3]_{p,q}[n]_{p,q}^2} \frac{1}{1+x^2} \\ &\leq 1 + C_1, \text{ for sufficiently large } n \end{aligned} \quad (20)$$

where  $C_1$  is a positive constant. From (20), there exists a positive constant  $K_1$  such that  $K_n(\varphi_x, p, q; x) \leq K_1(1+x^2)$ , for sufficiently large  $n$ .

Proceeding similarly  $\frac{1}{1+x^4} K_n(1+t^4, p, q; x) \leq 1 + C_2$ , for sufficiently large  $n$ , where  $C_2$  is a positive constant.

So there exists a positive constant  $K_2$  such that  $K_n(\varphi_x^2, p, q; x) \leq K_2(1+x^2)$ , where  $x \in [0, \infty)$   $n$  is large enough. Also we get

$$\begin{aligned} K_n(\psi_x^2, p, q; x) &= \left( \frac{p}{q^3} - \frac{2}{q} + 1 \right) x^2 + \left( \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} - \frac{2}{[2]_{p,q}[n]_{p,q}} \right) x + \frac{1}{[3]_{p,q}[n]_{p,q}^2} \\ &= \alpha_n x^2 + \beta_n x + \gamma_n. \end{aligned}$$

Hence from (19), we have

$$|K_n(f, p, q; x) - f(x)| \leq (1+x^2) \left( K_1 + \frac{1}{\delta_n} K_2 \sqrt{\alpha_n x^2 + \beta_n x + \gamma_n} \right) \Omega_2(f, \delta_n).$$

If we take  $\delta_n = \max\{\alpha_n, \beta_n, \gamma_n\}$ , then we get

$$\begin{aligned} |K_n(f, p, q; x) - f(x)| &\leq (1+x^2) \left( K_1 + K_2 \sqrt{x^2 + x + 1} \right) \Omega_2(f, \delta_n) \\ &\leq K_3(1+x^{2+\lambda}) \Omega_2(f, \delta_n), \text{ for sufficiently large } n \text{ and } x \in [0, \infty). \end{aligned}$$

Hence the proof is completed.

# References

- [1] T. Acar,  $(p, q)$ -generalization of Szász-Mirakjan operators, arXiv:submit/1263016[math.CA].
- [2] A. Aral, A generalization of Szász-Mirakjan operators based on  $q$ -integers, Math. Comput. Model. 47(9-10), (2008) 1052-1062.
- [3] Ali Aral and V. Gupta, The  $q$ -derivative and applications to  $q$ -Szász-Mirakjan operators, Calcolo 43(3), (2006) 151-170.
- [4] S.N. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités, Comm. Soc. Math. Kharkow (2), 13 (1912-1913) 1-2.
- [5] R. Chakrabarti and R. Jagannathan, A  $(p, q)$ -oscillator realization of two parameter quantum algebras, J. Phys. A: Math. Gen., 24 (1991) 711-718.
- [6] O. Doğru and E.A. Gadjieva, Agirlikli uzaylarda Szász tipinde operatörler dizisinin sürekli fonksiyonlara yaklasimi',II, Kizilirmak Uluslararası Fen Bilimleri Kongresi Bildiri Kitabı, 29-37, Kirikkale (1998) (Konusmacı: O. Dogru) (Türkçe olarak sunulmuş ve yayınlanmıştır).
- [7] A.D. Gadjieva, A problem on the convergence of a sequence of positive linear operators on unbounded sets, and theorems that are analogous to P.P. Korovkin's theorem, Dokl. Akad. Nauk SSSR 218 (1974), 1001-1004 (in Russian); Sov. Math. Dokl. 15 (1974), 1433-1436 (in English).
- [8] R. Jagannathan and K.S. Rao, Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, Proceedings of the International Conference on Number Theory and Mathematical Physics, December 2005, 20-21.
- [9] A.J. López-Moreno, Weighted simultaneous approximation with Baskakov type operators, Acta Math. Hungar. 104(1-2), (2004) 143-151.
- [10] A. Lupas, A  $q$ -analogue of the Bernstein operator, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, 9(1987), 85-92.
- [11] N.I. Mahmudov, On  $q$ -parametric Szász-Mirakjan operators, Mediterr. J. Math. 7(3) (2010), 297-311.
- [12] M. Mursaleen, K.J. Ansari and Asif Khan, On  $(p, q)$ -analogue of Bernstein operators, Appl. Math. Comput., doi: 10.1016/j.amc.2015.04.090.
- [13] M. Mursaleen, K.J. Ansari and Asif Khan, On  $(p, q)$ -analogue of Bernstein operators(Revised), arXiv:1503.07404v2 [math.CA] 20 Nov 2015.
- [14] M. Mursaleen, K.J. Ansari and Asif Khan, Some approximation results by  $(p, q)$ -analogue of Bernstein-Stancu operators, Appl. Math. Comput. 264 (2015) 392-402.
- [15] M. Mursaleen, Faisal Khan and Asif Khan, Approximation properties for Kings type modified  $q$ -BernsteinKantorovich operators, Math. Meth. Appl. Sci. 2015, doi: 10.1002/mma.3454.
- [16] G.M. Phillips, Bernstein polynomials based on the  $q$ -integers, The heritage of P.L.Chebyshev, Ann. Numer. Math., 4 (1997) 511-518.

- [17] P.N. Sadjang, On the fundamental theorem of  $(p, q)$ -calculus and some  $(p, q)$ -Taylor formulas, arXiv:1309.3934v1[math.QA].
- [18] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Nat. Bur. Stand. 45 (1950), 239-245.